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DIFFUSIVE APERTURE DUE TO LONG-RANGE BEAM-BEAM INTERACTION

Y. Papaphilippou, BNL, Upton NY 11973, USA, F. Zimmermann, CERN, Geneva, Switzerland

Abstract

Weak-strong tracking simulations for the Large Hadron Collider (LHC) show that long-range beam-beam collisions give rise to a well-defined diffusive aperture beyond which particles are lost quickly. We calculate the tune shift with amplitude and single-resonance driving terms for beam-beam collisions with an arbitrary transverse offset, and then apply Chirikov resonance overlap criterion to construct an analytical estimate of the diffusive aperture induced by long-range collisions. The analytical results are compared with tracking data for the LHC.

1 INTRODUCTION

In a colliding-beam storage ring one of the largest perturbations affecting the motion of beam particles is the collision with the opposing beam. This interaction occurs, unavoidably, in the form of head-on collisions between bunches of the two beams at a designated interaction point (IP) with minimum beta function. Many past studies for colliding proton beams have shown that simulations of head-on collisions can only reproduce the experimental data if a modulation of the betatron tune of the order of 10^{-4} is included, and that a transverse offset at the head-on collision point strongly enhances diffusion and particle losses.

Future colliders employ large trains of closely spaced bunches, and individual bunches encounter many bunches of the opposing beam at various long-range (l.r.) or 'parasitic' collision points, where the beams are not yet fully separated. The effect of the l.r. collisions depends on the ratio of the beam crossing angle to the rms beam divergence at the main interaction point, and also on the total number of parasitic collisions. In case of the LHC, a 7-TeV double-ring proton collider presently under construction, there are about 15 parasitic collision points on either side of the two main collision points.

Simulations predict that the l.r. collisions in the LHC give rise to a well defined border of stability at an amplitude, which we call the "diffusive aperture" [1,2]. As illustrated in Fig. 1, this diffusive aperture is insensitive to the presence of the head-on collision, and only marginally affected by transverse offsets or small tune ripple. Thus, previous studies for head-on collisions are not directly applicable, and a better understanding of the role of the l.r. interaction is called for.

In this report, we derive an analytical estimate of the diffusive aperture induced by the l.r. collisions. For this, we apply the Chirikov overlap criterion to a simplified 2D model of l.r. interactions in one IP of a circular machine, having the parameters of the LHC. The analytical result is compared with tracking simulations.

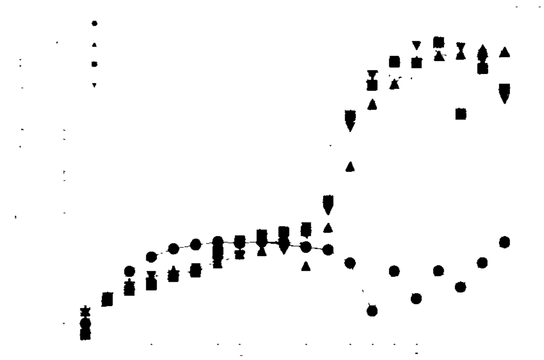

Figure 1: The change of action variance per turn as a function of starting amplitude in the LHC with $9.5\sigma_x'$ separation [1]. Whenever l.r. collisions are present, the diffusion rate increases sharply at about 6σ.

2 THEORY AND SIMULATIONS

Particle coordinate and slope at the IP are $x = -\sqrt{2J\beta^*}\sin\phi$, $x' = -\sqrt{2J/\beta^*}\cos\phi$, where (J,ϕ) are action-angle variables, and a prime denotes the derivative with respect to s. The l.r. collisions occur at a betatron phase advance close to $\pi/2$ from the IP. The collisions before and after the IP add up. Thus, the net effect of all l.r. collisions around one IP can be represented as a single shift in x. Considering only one primary IP, a full turn around the storage ring is described by

$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = \begin{pmatrix} \cos\mu & \beta^*\sin\mu \\ -\sin\mu/\beta^* & \cos\mu \end{pmatrix} \begin{pmatrix} x + f(x') \\ x' \end{pmatrix}_0$$

where θ_c is the full crossing angle, $f(x')$ the kick

$$f(x') = K\left[\frac{1}{x'+\theta_c} - \frac{1}{\theta_c}\right]\left(1 - e^{-\frac{(x'+\theta_c)^2}{2\sigma_{x'}^2}}\right)$$

and the constant $K = (-2r_pN_bn_{par}/\gamma)$, with N_b the bunch population and n_{par} the total number of l.r.-collision points. A static dipole kick was subtracted. Note that the negative sign of the coefficient K applies to equal-charge beams, as in the LHC.

If the oscillation amplitudes are small compared with the crossing angle, we can drop the exponential term, and the force decreases inversely with the distance to the other beam, $r' = (x' + \theta_c)$. Figure 2 compares the diffusive apertures simulated using the exact and the approximated l.r. beam-beam force as a function of bunch population. The figure demonstrates that for bunch intensities above 5×10^{11} the $1/r'$ approximation works well. For lower intensities, the $1/r'$ approximation gives a smaller diffusive aperture, and thus, it can be used as a "worst case scenario" estimate.

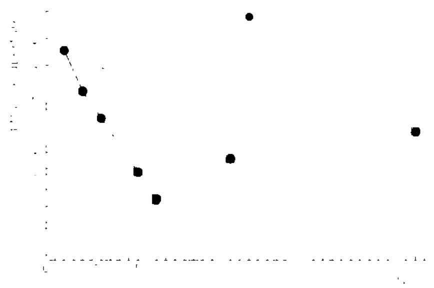

Figure 2: Simulated diffusive aperture as a function of bunch population, for 30 parasitic collision points and $\theta_c \approx 9.5\sigma_{x'}$. The figure compares the results for the exact force and those obtained using the $1/r'$ approximation.

The corresponding Hamiltonian simply consists of a periodic series of l.r. kicks and linear rotations $H(J,\phi,\theta) = Q_0 J + V(J,\phi)\sum_p e^{ip\theta}\frac{1}{2\pi}$ where Q_0 is the unperturbed tune. The beam-beam potential V can be written as

$$V(J,\phi) = \frac{K}{2}\left[-E_i\left(-\frac{(x'+\theta_c)^2}{2\sigma_{x'}^2}\right) + \ln\left(\frac{(x'+\theta_c)^2}{2\sigma_{x'}^2}\right) - \frac{2x'}{\theta_c}\left(1-e^{-\frac{\theta_c^2}{2\sigma_{x'}^2}}\right)\right]$$

where E_i is the exponential integral $E_i(u) = \int_{-\infty}^{u}\frac{e^{u'}}{u'}\,du'$. Near a resonance $nQ \approx p$ of order n, we write the Hamiltonian as $H_r \approx Q_0 J + g(J) + h_n\cos(n\phi - p\theta)$. Detuning with amplitude, dg/dJ and driving term h_n are calculated as $dg(J)/dJ = \langle\partial V/\partial J\rangle_\phi/(2\pi)$, and $h_n(J) = \frac{1}{2\pi^2}\int V(J,\phi)\cos n\phi\, d\phi$.

Calculation of the integrals yields

$$\frac{dg(J)}{dJ} = -\frac{K}{2\pi\beta^*\sqrt{\theta_c^2 - 2J/\beta^*}}\left[\frac{1}{\theta_c + \sqrt{\theta_c^2 - 2J/\beta^*}} + \frac{e^{-\frac{\theta_c^2}{2\sigma_{x'}^2} - \frac{J}{2\beta^*\sigma_{x'}^2}}}{4\sqrt{2J/\beta^*}}\sum_{k,l=-\infty}^{\infty} I_k\left(-\frac{J}{2\beta^*\sigma_{x'}^2}\right) I_l\left(-\frac{\theta_c}{\sigma_{x'}^2}\sqrt{\frac{2J}{\beta^*}}\right)\mathcal{D}_{k,l}(J)\right]$$

with

$$\mathcal{D}_{k,l}(J) = \mathcal{R}^{|2k+l+1|} + \mathcal{R}^{|2k+l-1|} + \mathcal{R}^{|2k-l+1|} + \mathcal{R}^{|2k-l-1|}$$

and

$$\mathcal{R}(J) = \frac{\sqrt{2J/\beta^*}}{\sqrt{\theta_c^2 - 2J/\beta^*} + \theta_c}.$$

The symbols I_n typically represent the modified Bessel functions of order n. The first term in the square brackets corresponds to the $1/r'$ approximation. Figure 3 shows that the $1/r'$ term diverges at amplitudes equal to the beam-beam separation, whereas the full expression becomes 0.

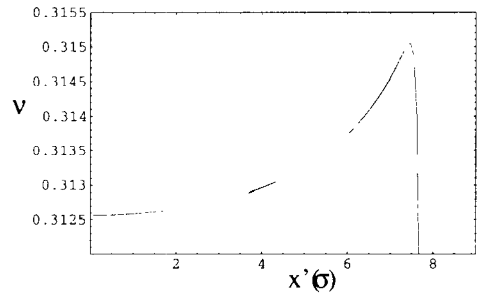

Figure 3: Tune as a function of amplitude in units of σ' due to l.r. beam-beam interaction for a separation of $9.5\sigma'_x$. Tracking result (green), the $1/r'$ term only (red), and the $1/r'$ part plus a single term ($k = 14$, $l = 3$) in the Bessel expansion (blue).

The agreement between the first order perturbation theory formula and simulation for the tune-shift is good.

The resonance driving term is

$$h_n = \frac{2K}{\pi n}\left[-\mathcal{R}^n + \frac{\sqrt{2J/\beta^*}e^{-\frac{\theta_c^2}{2\sigma_{x'}^2} - \frac{J}{2\beta^*\sigma_{x'}^2}}}{8\theta_c\sqrt{\theta_c - 2J/\beta^*}} \times \sum_{k,l=-\infty}^{\infty} I_k\left(-\frac{J}{2\beta^*\sigma_{x'}^2}\right) I_l\left(-\frac{\theta_c}{\sigma_{x'}^2}\sqrt{\frac{2J}{\beta^*}}\right)\mathcal{D}'_{k,l,n}\right]$$

with

$$\mathcal{D}'_{k,l,n}(J) = -\mathcal{R}^{|2k+l+n+1|} + \mathcal{R}^{|2k+l-n+1|} - \mathcal{R}^{|2k+l-n-1|} + \mathcal{R}^{|2k-l-n+1|} - \mathcal{R}^{|2k-l+n+1|} + \mathcal{R}^{|2k+l+n-1|} - \mathcal{R}^{|2k-l-n-1|} + \mathcal{R}^{|2k-l+n-1|}$$

Again, the first term in the square brackets represents the $1/r'$ force.

For simplicity, we now restrict the analysis to the $1/r'$ approximation. The resonance half-width of the nth order resonance is $\Delta J_{n,1/2} = 2\left(\frac{h_n}{d^2g/dJ^2}\right)^{1/2}$ and the distance between two resonances of order n_1 and n_2 is $\delta J = \left[\frac{1}{n_1} - \frac{1}{n_2}\right]\frac{1}{|d^2g/dJ^2|}$. Strong chaos exists when two resonances overlap [3]: $\frac{2}{3}\delta J \le \Delta J_{n_1,1/2} + \Delta J_{n_2,1/2}$. The factor $2/3$ accounts for the width of the separatrix and for higher-order islands [3,4]. This can be rewritten as

$$\frac{1}{3}\left(\frac{1}{n_1} - \frac{1}{n_2}\right) \le \sqrt{h_{n_1}d^2g/dJ^2} + \sqrt{h_{n_2}d^2g/dJ^2}$$

As an example, we consider $n_2 = (n_1 + 1)$, assume $h_{n_1} \approx h_{n_2}$, and insert the above expressions for g and h_n. The nonlinear equation describing the threshold of instability is

$$\left(\frac{1 + 2\sqrt{1-A^2}}{(1-A^2)^{3/2}(1+\sqrt{1-A^2})^2}\right) \times \left(\frac{A}{1+\sqrt{1-A^2}}\right)^{n_{\rm res}} \ge \frac{\pi^2\theta_c^4\beta^{*2}}{36K^2}\frac{1}{n_{\rm res}(n_{\rm res}+1)^2} \quad (1)$$

where $A \equiv \sqrt{2J/\beta^*}/\theta_c$ is the particle amplitude normalized to the separation, and $n_{\rm res} = n_1$. The overlap criterion is necessary, but not sufficient. In order to observe chaos, also resonances of order n_{res} need to be present near the threshold amplitude A. This depends on the nominal tune and on the detuning dg/dJ. With a working point at $Q_0 = 0.31$, the 3rd order resonance determines the position of the diffusive aperture, for $N_b \approx 5 \times 10^{11}$ and beyond, as shown in Fig. 4. Numerical solutions of inequality (1) are shown in Figs. 5a-c illustrating the dependence of the overlap amplitude on resonance order, bunch population, and crossing angle.

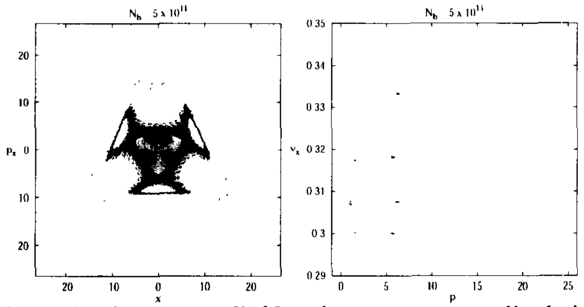

Figure 4: Phase space [left] and tune versus amplitude in units of σ from frequency map analysis [right] [1,5] for $N_b = 5 \times 10^{11}$ and 30 parasitic collision points.

At the nominal bunch population $N_b = 1.05 \times 10^{11}$, the diffusive aperture predicted by Eq. (1) seems to be rather insensitive to the resonance order. For large resonance orders or crossing angle, and also for small bunch populations, the onset of global chaos occurs at amplitudes where particles pass close to the center of the opposing beam. When the perturbation increases, resonances of lower order can induce global chaos at significantly smaller amplitudes. Ignoring the dependence on the right-hand side of Eq. (1), the diffusive aperture should increase roughly linearly with the crossing angle (see also Fig. 5c) and it should decrease approximately inversely with the square of the beta function at the IP. Unfortunately, the amplitude predicted by Eq. (1) considerably overestimates the diffusive aperture with respect to the simulations. This discrepancy may be due to our consideration of adjacent resonances, whereas in general the overlap can occur between resonances of any order.

Figure 5: Minimum amplitude at which the overlap condition Eq. (1) is fulfilled as a function of [top] resonance order n_{res} (for $\theta_c \approx 9.5\sigma'_x$, $N_b = 1.05 \times 10^{11}$), [center] bunch population (for the same crossing angle and five different resonance orders) and [bottom] crossing angle θ_c (for the nominal bunch population $N_b = 1.05 \times 10^{11}$ and four different resonance orders).

3 CONCLUSION

We have applied the Chirikov overlap criterion in order to arrive at an analytic estimate of the diffusive aperture due to the long-range beam-beam interaction in hadron colliders. For this, we have calculated analytical formulas for the tune-shift with amplitude and resonance driving terms, considering round beams and one-dimensional motion with time dependence, in the crossing plane only, through first order perturbation theory. Our formulas can easily be extended to the full transverse phase space. The analytical expression furnishes useful scaling laws for the dependence of this aperture on beam parameters, such as the crossing angle, the beta function at the IP or the bunch population. Finally, in order to improve the analytical estimate so as to become more consistent with the tracking results, the approximation of the $1/r'$ force and the restriction to adjacent resonances will have to be reviewed. These studies are still in progress.

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